Available online at www.sciencedirect.com
SCIENCE DIRECT®
ELSEVIER
Discrete Optimization 3 (2006) 111–122
DISCRETE OPTIMIZATION
www.elsevier.com/locate/disopt

Facets of the independent path-matching polytope

K.K.H. Cheung

School of Mathematics and Statistics, Carleton University, 1125 Colonel By Drive, Ottawa, ON K1S 5B6, Canada

Received 11 January 2005; received in revised form 25 January 2005; accepted 31 January 2006
Available online 22 March 2006

Abstract

Cunningham and Geelen introduced the independent path-matching problem as a common generalization of the weighted matching problem and the weighted matroid intersection problem. Associated with an independent path-matching is an independent path-matching vector. The independent path-matching polytope of an instance of the independent path-matching problem is the convex hull of all the independent path-matching vectors. Cunningham and Geelen described a system of linear inequalities defining the independent path-matching polytope. In this paper, we characterize which inequalities in this system induce facets of the independent path-matching polytope, generalizing previous results on the matching polytope and the common independent set polytope.

MSC: 90C27; 90C57

Keywords: Facets; Polytope; Matching; Matroid; Path-matching

1. Introduction

Cunningham and Geelen [2] introduced the notion of independent path-matchings as a common generalization of the weighted matching problem and the weighted matroid intersection problem. They defined an independent path-matching as follows. Let $G = (V, E)$ be a simple undirected graph without isolated vertices such that V is the disjoint union of T_1, T_2, and R, where T_1 and T_2 are stable sets. (A set of vertices is *stable* if no two vertices in the set are adjacent.) The sets T_1 and T_2 are called *terminal sets* of G and are sometimes denoted by $T_1(G)$ and $T_2(G)$. The set R is sometimes denoted by $R(G)$. Let M_1 and M_2 be loopless matroids on T_1 and T_2, respectively, with corresponding rank functions r_1 and r_2. An *independent path-matching* of (G, M_1, M_2) is a set $K \subseteq E$ such that every component of the graph $G[K]$ (the graph induced by the edge-set K) is a simple path from $T_1 \cup R$ to $T_2 \cup R$, all of whose internal vertices are in R and such that, for $i = 1$ and 2, the set of vertices of T_i in any of these paths is independent in M_i. An edge that is in a one-edge component in R is called a *matching edge* of K. Using the terminology of Frank and Szegő [6], the *value* of K is defined to be $|K| + |K'|$, where K' denotes the set of the matching edges of K.

A pair of subsets $D_1 \subseteq T_1 \cup R$, $D_2 \subseteq T_2 \cup R$ is called *stable* if no edge of G joins a vertex in $D_1 \backslash D_2$ to a vertex in D_2, or a vertex in $D_2 \backslash D_1$ to a vertex in D_1. Let $\mathrm{odd}(G)$ denote the number of components of G having an odd number

E-mail address: kcheung@math.carleton.ca.

1572-5286/$ - see front matter
doi:10.1016/j.disopt.2006.01.001

of vertices. For a subset $S \subseteq V$, let $G[S]$ denote the subgraph of G induced by S. Cunningham and Geelen [2] proved the following min–max formula:

Theorem 1. *The maximum value of an independent path-matching is equal to*

$$\min_{\text{stable}(D_1,D_2)} r_1(T_1\backslash D_1) + r_2(T_2\backslash D_2) + |R\backslash(D_1 \cup D_2)| + |R| - \text{odd}(G[D_1 \cap D_2]). \tag{1}$$

Theorem 1 generalizes the Tutte–Berge matching formula and Edmonds' matroid intersection min–max theorem.

A subset $X \subseteq V$ is called a *cut* separating the terminal sets T_1 and T_2 if, in the graph $G - X$, there is no path joining a vertex in $T_1\backslash X$ to a vertex in $T_2\backslash X$. For a subset $X \subseteq V$, let $\text{odd}_G(X)$ denote the number of odd components of $G - X$ disjoint from $T_1 \cup T_2$. Frank and Szegő [6] noted that Theorem 1 is equivalent to the following:

Theorem 2. *The maximum value of an independent path-matching is equal to*

$$|R| + \min_{X, a\ cut} (r_1(T_1 \cap X) + r_2(T_2 \cap X) + |R \cap X| - \text{odd}_G(X)). \tag{2}$$

Frank and Szegő [6] gave a purely combinatorial proof of Theorem 2 in the special case when M_1 and M_2 are free matroids. They remarked that the general case can be proved using similar arguments and standard techniques in matroid theory. (Cunningham and Geelen [2] gave two proofs of Theorem 1: one uses the Tutte-matrix and the other uses polyhedral methods.)

Given an independent path-matching K, the *independent path-matching vector* corresponding to K is the vector $\psi^K \in \mathbb{R}^E$ such that, for every $e \in E$,

$$\psi_e^K = \begin{cases} 1 & \text{if } e \in K\backslash K', \\ 2 & \text{if } e \in K', \\ 0 & \text{if } e \notin K \end{cases}$$

where K' is the set of matching edges of K. Let $\mathcal{K}^*(G, M_1, M_2)$ denote the set of independent path-matchings of (G, M_1, M_2). The *independent path-matching polytope* of (G, M_1, M_2), denoted by $\text{IP}(G, M_1, M_2)$, is the convex hull of $\{\psi^K : K \in \mathcal{K}^*(G, M_1, M_2)\}$. Cunningham and Geelen [2] proved the following:

Theorem 3. $\text{IP}(G, M_1, M_2)$ *is the set of all* $x \in \mathbb{R}^E$ *satisfying:*

$$x(\delta(v)) \le 2 \quad (v \in R) \tag{3}$$

$$x(\gamma(S)) \le |S| - 1 \quad (S \subseteq R, |S| \text{ odd}) \tag{4}$$

$$x(\gamma(S)) \le |S \cap R| \quad (i \in \{1, 2\}, T_i \subset S \subseteq T_i \cup R) \tag{5}$$

$$x(\delta(A)) \le r_i(A) \quad (i \in \{1, 2\},\ A \subseteq T_i) \tag{6}$$

$$x \ge 0. \tag{7}$$

Here, $\delta(S)$ denotes the set of edges with exactly one end in S, and $\gamma(S)$ denotes the set of edges with both ends in S. For convenience, $\delta(\{v\})$ is abbreviated as $\delta(v)$. We call inequalities (3) *degree inequalities*, inequalities (4) *blossom inequalities*, inequalities (5) *cap inequalities*, inequalities (6) *rank inequalities*, and inequalities (7) *non-negativity inequalities*.

As mentioned in [2], specializing Theorem 3 to the case when $T_1 = T_2 = \emptyset$ gives the following classical result due to Edmonds [4]:

Theorem 4. *Let* $G = (V, E)$ *be a graph. Let* $\text{MP}(G)$ *denote the matching polytope of* G*; that is, the convex hull of incidence vectors of matchings of* G*. Then* $\text{MP}(G)$ *is the set of all* $x \in \mathbb{R}^E$ *satisfying:*

$$x(\delta(v)) \le 1 \quad (v \in R)$$

$$x(\gamma(S)) \le (|S| - 1)/2 \quad (S \subseteq R,\ |S| \text{ odd})$$

$$x \ge 0.$$

(Theorem 3 in fact gives a description by linear inequalities of *twice* $\text{MP}(G)$.)

Also, specializing Theorem 3 to the case when $R = \emptyset$ and G consists of a perfect matching joining T_1 to T_2 gives another classical result that is also due to Edmonds [5]:

Theorem 5. *Let* $\mathrm{CP}(M_1, M_2)$ *denote the common independent set polytope of* M_1 *and* M_2*; that is, the convex hull of incidence vectors of common independent sets of two matroids* M_1, M_2 *on* E *with rank functions* r_1, r_2*, respectively. Then* $\mathrm{CP}(M_1, M_2)$ *is the set of all* $x \in \mathbb{R}^E$ *satisfying:*

$$x(A) \le r_1(A) \quad (A \subseteq E)$$
$$x(A) \le r_2(A) \quad (A \subseteq E)$$
$$x \ge 0.$$

Cunningham and Geelen [2] also showed that the system of Theorem 3 is totally dual integral (TDI), generalizing previous results for the matching polytope [3] and the common independent set polytope [5]. (A system $Ax \le b$ of linear inequalities is *totally dual integral* if, for every integral vector c for which the problem $\min\{b^T y : A^T y = c, y \ge 0\}$ has an optimal solution, it has an optimal solution that is integral.)

Pulleyblank and Edmonds [8] characterized the facets of $\mathrm{MP}(G)$, proving the following (see also p. 446 of [9] for a short proof):

Theorem 6. *Let* $G = (V, E)$ *be a graph. Let* $I = \{v \in V : \deg(v) \ge 3$*, or* $\deg(v) = 2$ *and* v *is contained in no triangle, or* $\deg(v) = 1$ *and the neighbour of* v *also has degree 1*$\}$*, and* $\mathcal{B} = \{U \subseteq V : |U| \ge 3$ *is odd,* $G[U]$ *is factor-critical and 2-connected*$\}$*. (A graph* G *is factor-critical if* $G - v$ *has a perfect matching for each vertex* v *of* G*.) Then*

(i) $x(\delta(v)) \le 1$ *is facet-inducing if and only if* $v \in I$*;*
(ii) $x(\gamma(S)) \le (|S| - 1)/2$ *is facet-inducing if and only if* $S \in \mathcal{B}$*;*
(iii) $x_e \ge 0$ *is facet-inducing for all* $e \in E$*.*

Giles [7] characterized which inequalities in Theorem 5 are facet-inducing. If $\mathrm{CP}(M_1, M_2)$ is full-dimensional, then the inequality $x_e \ge 0$ is facet-inducing for every $e \in E$. Giles proved the following for the other inequalites (see also p. 718 of [9] for a short proof):

Theorem 7. *Let* M_1 *and* M_2 *be loopless matroids on* E *with rank functions* r_1 *and* r_2*, respectively. For* $U \subseteq E$*, define* $r(U) := \min\{r_1(U), r_2(U)\}$*. Then, for* $U \subseteq E$*, the inequality*

$$x(U) \le r(U)$$

is facet-inducing for $\mathrm{CP}(M_1, M_2)$ *if and only if* U *cannot be partitioned into non-empty proper subsets* S_1, S_2 *with*

$$r(U) \ge r(S_1) + r(S_2)$$

and there is no proper superset U' *of* U *with* $r(U') \le r(U)$*.*

The goal of this paper is to give a common generalization of Theorems 6 and 7 by characterizing which of the inequalities in Theorem 3 are facet-inducing.

The rest of this section is spent on notation and some facts in polyhedral theory used in this paper. We refer the readers to [1] for basic results in matroid theory and polyhedral theory.

Let $G = (V, E)$ be a graph and $S \subseteq V$. The set of vertices in $V \backslash S$ adjacent to some vertex in S is denoted by $N_G(S)$ (or simply $N(S)$ if the graph in context is clear). For convenience, $N_G(\{v\})$ and $N(\{v\})$ are abbreviated as $N_G(v)$ and $N(v)$, respectively.

Let E be a finite set and let S be a subset of E. Denote the incidence vector of $e \in E$ by χ^e. For a vector $a \in \mathbb{R}^E$, define $\mathrm{supp}(a) := \{e \in E : a_e \ne 0\}$. Define $a|_S$ to be the projection of a onto $\mathbb{R}^S$. Let $P \subset \mathbb{R}^E$ be a polytope. Denote the projection of P onto S by $P|_S$; that is, $P|_S := \{x|_S : x \in P\}$. An inequality $a^T x \le b$ is *non-trivial* if $a \ne 0$. A subset $F \subseteq P$ is called a *face* of P if $F = \{x \in P : a^T x = b\}$ for some inequality $a^T x \le b$ valid for P. We say $a^T x \le b$ *induces* the face F. A *facet* of P is a face of P having dimension $\dim(P) - 1$. A valid inequality is *facet-inducing* if it induces a facet.

Let M be a matroid on E and let r be its rank function. Let $S \subseteq E$. The matroid obtained from deleting the elements in S is denoted by $M \backslash S$. The matroid obtained from contracting the elements in S is denoted by M/S. We say that S is *closed* (with respect to M) if $r(U) > r(S)$ for every proper superset U of S. For convenience, $M \backslash \{v\}$ and $M/\{v\}$

are written as $M\backslash v$ and M/v, respectively. Let N be a matroid on F such that $E \cap F = \emptyset$. The union of M and N is denoted by $M \oplus N$.

We now state some useful results in polyhedral theory.

Theorem 8. *Let P be a full-dimensional polytope. Any two inequalities that induce the same facet of P are positive scalar multiples of each other.*

An immediate corollary is the following:

Corollary 9. *Let $P := \{x : a_i^T x \le b_i,\ i = 1, \dots, m\}$ be a full-dimensional polytope with $a_i \neq 0$ for all $i = 1, \dots, m$. Let $a^T x \le b$ be a valid inequality for P. If, for some integer $k > 0$, there exist $\lambda_1, \dots, \lambda_k > 0$ and $\{i_1, \dots, i_k\} \subseteq \{1, \dots, m\}$ such that*

$$\sum_{j=1}^{k} \lambda_j a_{i_j} = a, \quad \sum_{j=1}^{k} \lambda_j b_{i_j} \le b$$

and a is not a positive scalar multiple of a_{i_j} for every $j \in \{1, \dots, k\}$, then $a^T x \le b$ is not facet-inducing.

Proof. Suppose that $a^T x \le b$ is facet-inducing and such λs exist. Let F be the facet induced by $a^T x \le b$. Then, for any $\bar{x} \in F$,

$$b = a^T \bar{x} = \sum_{j=1}^{k} \lambda_j a_{i_j}^T \bar{x} \le \sum_{j=1}^{k} \lambda_j b_{i_j} \le b.$$

Hence, equality holds throughout, implying that F is contained in the face induced by $a_{i_j}^T x \le b_{i_j}$ for every $j \in \{1, \dots, k\}$. As $a_i \neq 0$ for all i, it follows that, for every $j \in \{1, \dots, k\}$, $a_{i_j}^T x \le b_{i_j}$ is facet-inducing and thus, by Theorem 8, is a positive scalar multiple of $a^T x \le b$, which is a contradiction. □

Corollary 10. *Let $P \subset \mathbb{R}^E$ be a full-dimensional polytope. Let $a^T x \le b$ and $c^T x \le d$ be non-trivial valid inequalities for P such that $\{x \in P : a^T x = b\} \subseteq \{x \in P : c^T x = d\}$. Let $S = \operatorname{supp}(a)$ and $F = \{x \in P : a^T x = b\}$. If $\operatorname{supp}(c) \subseteq S$ and $F|_S$ is a facet of $P|_S$, then $c^T x \le d$ is a positive scalar multiple of $a^T x \le b$.*

Proof. Let $F' = F|_S$ and $P' = P|_S$. Since $a_e = c_e = 0$ for all $e \notin S$ and $c \neq 0$, both $a^T x \le b$ and $c^T x \le d$ induce the facet F' of P'. Since P' is a full-dimensional polytope in $\mathbb{R}^S$, the result follows from Theorem 8. □

2. Facet-inducing inequalities

Let $P := \{x \in \mathbb{R}^E : a_i^T x \le b_i,\ i = 1, \dots, m\}$ be a full-dimensional polytope. Given $i \in \{1, \dots, m\}$, three common methods for showing that $a_i^T x \le b_i$ is facet-inducing are:

(1) display $|E|$ affinely independent vectors in P that satisfy $a_i^T x \le b_i$ with equality;
(2) show that any non-trivial inequality $c^T x \le d$ valid for P such that $\{x \in P : a_i^T x = b_i\} \subseteq \{x \in P : c^T x = d\}$ must be a positive scalar multiple of $a_i^T x \le b_i$;
(3) exhibit a point that violates $a_i^T x \le b_i$ but satisfies the remaining inequalities defining P.

We will use each of the above methods in this paper to determine the facet-inducing inequalities for $\mathrm{IP}(G, M_1, M_2)$.

2.1. Non-negativity inequalities

Since M_1 and M_2 are loopless, $\{e\}$ is an independent path-matching of (G, M_1, M_2) for every $e \in E$. Since $0 \in \mathrm{IP}(G, M_1, M_2)$, $\mathrm{IP}(G, M_1, M_2)$ is full-dimensional and the next result is immediate.

Theorem 11. *The inequality $x_e \ge 0$ induces a facet of $\mathrm{IP}(G, M_1, M_2)$ for all $e \in E$.*

Proof. The zero vector and the vectors $\psi^{\{f\}}$ for all $f \in E\backslash e$ give $|E|$ affinely independent vectors that satisfy $x_e \ge 0$ with equality. □

2.2. Degree inequalities

Theorem 12. *The inequality $x(\delta(v)) \le 2$ is facet-inducing if and only if one of the following holds:*

(i) $|N(v) \cap R| + \min\{1, |N(v) \cap T_1|\} + \min\{1, |N(v) \cap T_2|\} \ge 3$;
(ii) $N(v) \cap R = \{u\}$ *and, for some* $i \in \{1, 2\}$, $N(u) \cap T_i = \emptyset$, $N(v) \cap T_i \ne \emptyset$, *and* $N(v) \cap T_{3-i} = \emptyset$;
(iii) $N(v) = \{u, w\} \subseteq R$ *and* $uw \notin E$;
(iv) $N(v) = \{u\} \subseteq R$ *with* $\deg(u) = 1$.

Proof. We first show necessity. Suppose none of (i)–(iv) is true. Then $|N(v) \cap R| + \min\{1, |N(v) \cap T_1|\} + \min\{1, |N(v) \cap T_2|\} \le 2$ and we are in one of the following cases:

Case 1: $N(v) \cap R = \emptyset$.

If $\min\{1, |N(v) \cap T_1|\} + \min\{1, |N(v) \cap T_2|\} = 1$, then $N(v) \cap T_i \ne \emptyset$ and $N(v) \cap T_{3-i} = \emptyset$ for some $i \in \{1, 2\}$. Hence, $\delta(v) = \gamma(\{v\} \cup T_i)$ and the cap inequality $x(\gamma(\{v\} \cup T_i)) \le 1$ shows that $x(\delta(v)) \le 2$ induces the empty face. If $\min\{1, |N(v) \cap T_1|\} + \min\{1, |N(v) \cap T_2|\} = 2$, then $N(v) \cap T_1 \ne \emptyset$ and $N(v) \cap T_2 \ne \emptyset$ and $x(\delta(v)) \le 2$ is the sum of the cap inequalities $x(\gamma(\{v\} \cup T_1)) \le 1$ and $x(\gamma(\{v\} \cup T_2)) \le 1$. By Corollary 9, $x(\delta(v)) \le 2$ is not facet-inducing.

Case 2: $N(v) \cap R = \{u\}$ and, for some $i \in \{1, 2\}$, $N(u) \cap T_i \ne \emptyset$, $N(v) \cap T_i \ne \emptyset$, and $N(v) \cap T_{3-i} = \emptyset$.

Let $S = \{u, v\} \cup T_i$. Then $\gamma(S) \backslash \delta(v) \ne \emptyset$, and $x(\delta(v)) \le 2$ is the sum of the cap inequality $x(\gamma(S)) \le 2$ and $-x_e \le 0$ for all $e \in \gamma(S) \backslash \delta(v)$. By Corollary 9, $x(\delta(v)) \le 2$ is not facet-inducing.

Case 3: $N(v) = \{u, w\} \subseteq R$ and $uw \in E$.

Then $x(\delta(v)) \le 2$ is the sum of the blossom inequality $x(\gamma(\{u, v, w\})) \le 2$ and the inequality $-x_{uw} \le 0$. By Corollary 9, $x(\delta(v)) \le 2$ is not facet-inducing.

Case 4: $N(v) = \{u\} \subseteq R$ and $\deg(u) > 1$.

Then $x(\delta(v)) \le 2$ is the sum of the degree inequality $x(\delta(u)) \le 2$ and the inequalities $-x_e \le 0$ for all $e \in \delta(u) \backslash \{uv\}$. By Corollary 9, $x(\delta(v)) \le 2$ is not facet-inducing.

We now show sufficiency by constructing $|E|$ affinely independent vectors that satisfy $x(\delta(v)) \le 2$ with equality. First, note that $\delta(v) \cap \gamma(R) \ne \emptyset$. For each $e \in \delta(v) \cap \gamma(R)$, let $K_e = \{e\}$; for each $e \in \delta(v) \backslash \gamma(R)$, let $K_e = \{e, f\}$, where f is an edge in $\delta(v) \cap \gamma(R)$; for each $e \in E \backslash \delta(v)$, if $N(v) \cap R = \{u\}$ and e is incident with u, then let $K_e = \{e, uv, vw\}$ for some $w \in N(v) \cap (T_1 \cup T_2)$ such that K_e is an independent path-matching; otherwise, let $K_e = \{e, f\}$, where f is an edge in $\delta(v) \cap \gamma(R)$ that does not meet e. Note that each ψ^{K_e} satisfies $x(\delta(v)) \le 2$ with equality and $\{\psi^{K_e}, e \in E\}$ is an affinely independent set. The result now follows. □

2.3. Blossom inequalities

These inequalties come from the blossom inequalities for the matching polytope and the results there carry over to the current setting. We first prove an elementary lemma that will also be used when we consider the rank inequalities.

Lemma 13. *Let (G, M_1, M_2) and (G', M_1', M_2') be instances of the independent path-matching problem such that G' is a subgraph of G, $T_1(G') = T_1(G) \cap V(G')$, $T_2(G') = T_2(G) \cap V(G')$, $M_1' = M_1 \backslash (T_1(G) \backslash T_1(G'))$, and $M_2' = M_2 \backslash (T_2(G) \backslash T_2(G'))$. Then $\mathrm{IP}(G, M_1, M_2)|_{E(G')} = \mathrm{IP}(G', M_1', M_2')$.*

Proof. Let $P = \mathrm{IP}(G', M_1', M_2')$ and $Q = \mathrm{IP}(G, M_1, M_2)|_{E(G')}$. Clearly, every independent path-matching of (G', M_1', M_2') is also an independent path-matching of (G, M_1, M_2). Hence, $P \subseteq Q$.

We now show that $Q \subseteq P$. Let $x \in Q$. Let $y \in \mathrm{IP}(G, M_1, M_2)$ be such that $x = y|_{E(G')}$. Let $K_1, \ldots, K_m$ be independent path-matchings of (G, M_1, M_2) such that

$$y = \sum_{i=1}^{k} \lambda_i \psi^{K_i}$$

for some $\lambda_1, \ldots, \lambda_m > 0$ with $\sum_{i=1}^{m} \lambda_i = 1$. To show that $x \in P$, it suffices to show that $\psi^{K_i}|_{E(G')} \in P$ for $i = 1, \ldots, m$. For $i \in \{1, \ldots, m\}$, $K_i' := K_i \cap E(G')$ is an independent path-matching of (G', M_1', M_2'). Furthermore, for any $e \in E(G')$, either $\psi_e^{K_i'} = \psi_e^{K_i}$ or $\psi_e^{K_i'} = 2$ and $\psi_e^{K_i} = 1$. Let $S_i = \{e \in E(G') : \psi_e^{K_i'} \ne \psi_e^{K_i}\}$. Since $K_i' \backslash S_i$ is an independent path-matching of (G', M_1', M_2') and every edge in S_i is a matching edge of K_i', we have $\psi^{K_i}|_{E(G')} = \frac{1}{2} \psi^{K_i'} + \frac{1}{2} \psi^{K_i' \backslash S_i} \in P$, as desired. □

Theorem 14. *The inequality $x(\gamma(S)) \leq |S| - 1$ is facet-inducing if and only if $|S| \geq 3$ is odd, $G[S]$ is factor-critical and 2-connected.*

Proof. Let $\mathcal{B} = \{S \subseteq V : |S| \geq 3$ is odd, $G[S]$ is factor-critical and 2-connected$\}$. Let $Q = \mathrm{IP}(G, M_1, M_2)|_{\gamma(S)}$. By Lemma 13, we see that $Q = 2\mathrm{MP}(G[S])$. Hence, necessity follows from the fact that $S \in \mathcal{B}$ is necessary for $x(\gamma(S)) \leq (|S| - 1)/2$ to be facet-inducing for $\mathrm{MP}(G[S])$ (Theorem 6). In fact, if $S \notin \mathcal{B}$, the inequality $x(\gamma(S)) \leq |S| - 1$ either induces the empty face or can be written as an integer linear combination of non-negativity inequalities, degree inequalities, and other blossom inequalities. This follows from the fact that the facet-inducing inequalities of Theorem 6 give a TDI system (see [3]).

To prove sufficiency, let F denote the face of $\mathrm{IP}(G, M_1, M_2)$ induced by $x(\gamma(S)) \leq |S| - 1$. Let $c^T x \leq d$ be a non-trivial valid inequality satisfied with equality by every $x \in F$. We show that $c^T x \leq d$ must be a positive scalar multiple of $x(\gamma(S)) \leq |S| - 1$. We claim that $c_e = 0$ for all $e \in E\backslash\gamma(S)$. Let $e \in E\backslash\gamma(S)$. Since $G[S]$ is factor-critical, there exists a matching K of $G[S]$ of cardinality $(|S| - 1)/2$ such that $K \cup \{e\}$ is a matching of G. As M_1 and M_2 are loopless matroids, $K \cup \{e\}$ is an independent path-matching of (G, M_1, M_2). Clearly, both $\psi^K, \psi^{K\cup\{e\}}$ satisfy $x(\gamma(S)) \leq |S| - 1$ with equality. Therefore, $\psi^K, \psi^{K\cup\{e\}} \in F$, implying that $c_e = 0$. This proves our claim. As $x(\gamma(S)) \leq |S| - 1$ induces a facet of Q (Theorem 6), by Corollary 10, $c^T x \leq d$ is a positive scalar multiple of $x(\gamma(S)) \leq |S| - 1$, as desired. □

2.4. Cap inequalities

We first state a lemma that follows immediately from Theorem 2.

Lemma 15. *If $r_1(T_1 \cap X) + r_2(T_2 \cap X) + |R \cap X| - \mathrm{odd}_G(X) \geq 1$ for every cut X, then any independent path-matching of maximum value must contain a path joining a vertex in T_1 and a vertex in T_2.*

Theorem 16. *Let $i \in \{1, 2\}$ and let S be such that $T_i \subset S \subseteq T_i \cup R$. The inequality*

$$x(\gamma(S)) \leq |S \cap R| \tag{8}$$

is facet-inducing if and only if $G[S \cap R]$ is connected and one of the following holds:

(i) *$S \cap R = \{u\}$ such that $N(u) \cap T_i \neq \emptyset$ and, for each $A \subseteq T_i$ with $N(u) \cap T_i \subseteq A$, either $r_i(A) > 1$ or $\delta(A) = \gamma(S)$;*
(ii) *$|S \cap R| = 2$ such that, for any $v \in S \cap R$, $\gamma(S)$ is not a proper subset of $\delta(v)$;*
(iii) *$|S \cap R| \geq 3$ and all of the following hold:*
  (a) *$N(S \cap R) \cap T_i \neq \emptyset$;*
  (b) *$r_i(T_i \cap X) + |R \cap X| - \mathrm{odd}_{G[S]}(X) \geq 1$ for every non-empty $X \subseteq S$;*
  (c) *$G[S] - v$ does not have a component disjoint from T_i for every $v \in S \cap R$.*

Proof. By symmetry, we may assume that $i = 1$. We first show that it is necessary that $G[S \cap R]$ is connected. Suppose not. Let k denote the number of components of $G[S \cap R]$ and let $V_1, \ldots, V_k$ be the vertex-sets of the components. If $N(V_j) \cap T_1 \neq \emptyset$ for all $j \in \{1, \ldots, k\}$, then $\gamma(S) \neq \gamma(V_j \cup T_1)$ for all $j \in \{1, \ldots, k\}$. Adding the cap inequalities $x(\gamma(V_j \cup T_1)) \leq |V_j|$ for all $j = 1, \ldots, k$ gives (8). By Corollary 9, (8) is not facet-inducing. Suppose $N(V_j) \cap T_1 = \emptyset$ for some j. If there exists such j for which $|V_j|$ is odd, then adding the blossom inequality $x(\gamma(V_j)) \leq |V_j| - 1$ and the cap inequality $x(\gamma(S\backslash V_j)) \leq |(S\backslash V_j) \cap R|$ gives $x(\gamma(S)) \leq |S \cap R| - 1$, implying that (8) induces the empty face. Otherwise, pick any j such that $N(V_j) \cap T_1 = \emptyset$. Let $w \in V_j$ and $W = V_j\backslash\{w\}$. Adding the degree inequality $x(\delta(w)) \leq 2$, the blossom inequality $x(\gamma(W)) \leq |W| - 1$, the cap inequality $x(\gamma(S\backslash V_j)) \leq |(S\backslash V_j) \cap R|$, and the inequalities $-x_e \leq 0$ for all $e \in \delta(w)\backslash\gamma(V_j)$ gives (8). Observe that none of $\delta(w)$, $\gamma(W)$, and $\gamma(S\backslash V_j)$ are equal to $\gamma(S)$. By Corollary 9, (8) is not facet-inducing.

We now assume that $G[S \cap R]$ is connected and consider three cases.

Case 1: $|S \cap R| = 1$.

Let u denote the only vertex in $S \cap R$. Suppose that (i) does not hold. If $N(u) \cap T_1 = \emptyset$, then (8) induces the empty face. If $N(u) \cap T_1 \neq \emptyset$, then there exists $A \subseteq T_1$ with $N(u) \cap T_1 \subseteq A$ such that $r_1(A) = 1$ and $\delta(A)\backslash\gamma(S) \neq \emptyset$. Adding the rank inequality $x(\delta(A)) \leq r_1(A)$ and the inequalities $-x_e \leq 0$ for all $e \in \delta(A)\backslash\gamma(S)$ gives (8). By Corollary 9, (8) is not facet-inducing. Conversely, suppose that (i) holds. Then $|E|$ affinely independent vectors satisfying (8) with equality can be constructed as follows. For each $e \in \gamma(S)$, let $K_e = \{e\}$; for each $e \in E \backslash \gamma(S)$, we claim that there exists $f \in \gamma(S)$ such that $K_e = \{e, f\}$ is an independent path-matching. Clearly, such an edge f always exists if $e \notin \delta(T_1)$. If $e \in \delta(T_1)$, then the claim follows from the fact that $r_1(\{w\} \cup N(u) \cap T_1) > 1$, where w is the end of e in T_1. Note that each ψ^{K_e} satisfies (8) with equality and that $\{\psi^{K_e} : e \in E\}$ is an affinely independent set. Hence, (8) is facet-inducing.

Case 2: $|S \cap R| = 2$.

Suppose that (ii) does not hold. Let $v \in S \cap R$ be such that $\gamma(S)$ is a proper subset of $\delta(v)$. Adding the degree inequality $x(\delta(v)) \leq 2$ and the inequalities $-x_e \leq 0$ for all $e \in \delta(v) \backslash \gamma(S)$ gives (8). By Corollary 9, (8) is not facet-inducing. Conversely, suppose that (ii) holds. Let $S \cap R = \{u, v\}$. If $N(\{u, v\}) \cap T_1 = \emptyset$, then $N(u) = \{v\}$ and $N(v) = \{u\}$, since $G[S \cap R]$ is connected. By (iv) of Theorem 12, (8) is facet-inducing. Suppose that $N(\{u, v\}) \cap T_1 \neq \emptyset$. If $N(u) \cap T_1 = \emptyset$, say, then $\delta(v) = \gamma(S)$. By (ii) of Theorem 12, (8) is facet-inducing. If $N(w) \cap T_1 \neq \emptyset$ for all $w \in \{u, v\}$, then let $u' \in N(u) \cap T_1$, $v' \in N(v) \cap T_1$ and $\hat{x} \in \mathbb{R}^E$ be such that

$$\hat{x}_e = \begin{cases} \dfrac{1}{2} & \text{if } e \in \{uu', vv'\}; \\ \dfrac{3}{2} & \text{if } e = uv; \\ 0 & \text{otherwise.} \end{cases}$$

It is easy to see that $\hat{x}$ violates (8) but satisfies all the other cap inequalities and all the degree inequalities, blossom inequalities, and rank inequalities. Hence, (8) is facet-inducing.

Case 3: $|S \cap R| \geq 3$.

Suppose that (a) does not hold. If $|S \cap R|$ is odd, then the blossom inequality $x(\gamma(S \cap R)) \leq |S \cap R| - 1$ shows that (8) induces the empty face. Suppose that $|S \cap R|$ is even. As $|S \cap R| \geq 3$, there exists $w \in S \cap R$ such that $\gamma(S) \backslash \delta(w) \neq \emptyset$. Let $W = (S \backslash \{w\}) \cap R$. Adding the degree inequality $x(\delta(w)) \leq 2$, the blossom inequality $x(\gamma(W)) \leq |W| - 1$, and the inequalities $-x_e \leq 0$ for all $e \in \delta(w) \backslash \gamma(S)$ gives (8). As $\gamma(S) \neq \delta(w)$ and $\gamma(S) \neq \gamma(W)$, (8) is not facet-inducing by Corollary 9.

Now, suppose that (a) holds but (b) does not. Choose $X \neq \emptyset$ such that $r_1(T_1 \cap X) + |R \cap X| - \mathrm{odd}_{G[S]}(X) \leq 0$. Observe that $m := \mathrm{odd}_{G[S]}(X) \geq 1$. Let $O_1, \ldots, O_m$ denote the vertex-sets of the odd components of $G[S] - X$ disjoint from T_1. Let $O = O_1 \cup \cdots \cup O_m$ and $S' = S \backslash (O \cup (R \cap X))$. Adding the blossom inequalities $x(\gamma(O_i)) \leq |O_i| - 1$ for all $i = 1, \ldots, m$, the degree inequalities $x(\delta(v)) \leq 2$ for all $v \in R \cap X$, the rank inequality $x(\delta(T_1 \cap X)) \leq r_1(T_1 \cap X)$, the cap inequality $x(\gamma(S')) \leq |S' \cap R|$ and the inequalities $-x_e \leq 0$ for all $e \in \gamma(X) \cup (\delta(X) \cap \delta(S)) \cup (\delta(T_1 \cap X) \cap \gamma(S'))$ gives $x(\gamma(S)) \leq d$, where

$$\begin{aligned} d &= \sum_{i=1}^{m} (|O_i| - 1) + 2|R \cap X| + r_1(T_1 \cap X) + |S' \cap R| \\ &= r_1(T_1 \cap X) + |R \cap X| - \mathrm{odd}_{G[S]}(X) + |O| + |R \cap X| + |S' \cap R| \leq |S \cap R|. \end{aligned}$$

If $\gamma(S) \subseteq \delta(v)$ for some $v \in S \cap R$, then we could have $X = \{v\}$ and $|O_i| = 1$ for $i = 1, \ldots, m$ and $S' \cap R = \emptyset$. Hence, $d = 2 < |S \cap R|$, implying that (8) induces the empty face. Suppose that $\gamma(S) \backslash \delta(v) \neq \emptyset$ for all $v \in S \cap R$. Clearly, $\gamma(S) \neq \gamma(S')$ and $\gamma(S) \neq \gamma(O_i)$ for all $i = 1, \ldots, m$. Note that $\gamma(S) \neq \delta(T_1 \cap X)$. (This is obvious if $\gamma(S \cap R) \neq \emptyset$. If $\gamma(S \cap R) = \emptyset$, then, as $G[S \cap R]$ is connected, $|S \cap R| = 1$, contradicting our assumption.) It follows from Corollary 9 that (8) is not facet-inducing.

Now, suppose that (a) and (b) hold but (c) does not hold for some $v \in R$. Let $V_1, \ldots, V_k$ be the vertex-sets of the components of $G[S] - v$ disjoint from T_1. As (b) holds, $|V_j|$ is even for all $j = 1, \ldots, k$. Let $S' = S \backslash (V_1 \cup \cdots \cup V_k)$ and $W = \{v\} \cup V_1 \cup \cdots \cup V_k$. Notice that $|W|$ is odd and that $|S' \cap R| + |W| = |S \cap R| + 1$. Adding the blossom inequality $x(\gamma(W)) \leq |W| - 1$ and the cap inequality $x(\gamma(S')) \leq |S' \cap R|$ gives $x(\gamma(S)) \leq |S \cap R|$. As (a) holds, $\gamma(S)$ is equal to neither $\gamma(S')$ nor $\gamma(W)$. By Corollary 9, (8) is not facet-inducing.

We now assume that (iii) holds and show that (8) is facet-inducing. Let $\mathcal{F}$ be the set of independent path-matching vectors that satisfy (8) with equality. Let

$$\sum_{e \in E} c_e x_e \leq d \tag{9}$$

be a non-trivial valid inequality that is also satisfied by every element in $\mathcal{F}$ with equality. We show that (9) must be a positive scalar multiple of (8).

We first show that $c_e = 0$ for every $e \in E \backslash \gamma(S)$. Let $e \in E \backslash \gamma(S)$. Suppose that $e \notin \delta(S)$. Apply Theorem 2 to (G', M_1', M_2'), where $G' = G[S]$, $T_1(G') = T_1$, $T_2(G') = \emptyset$, $M_1' = M_1$, and M_2' is the empty matroid. By (b), we see that there exists an independent path-matching K of (G', M_1', M_2') having value $|R(G')| = |S \cap R|$. Note that K and $K \cup \{e\}$ are independent path-matchings of (G, M_1, M_2) with $\psi^K, \psi^{K \cup \{e\}} \in \mathcal{F}$ and $\psi^{K \cup \{e\}} = \psi^K + 2\chi^e$. As both ψ^K and $\psi^{K \cup \{e\}}$ must satisfy (9) with equality, it follows that $c_e = 0$. Now, suppose $e = uv \in \delta(S)$ with $u \in S$ and $v \notin S$. Consider the independent path-matching problem on (G', M_1', M_2'), where $G' = (S \cup \{v\}, \gamma(S) \cup \{uv\})$, $T_1(G') = T_1$, $T_2(G') = \{v\}$, $M_1' = M_1$ and M_2' is the free matroid. Let r_i' denote the rank function of M_i' for $i = 1, 2$. Let X' be a cut of G'. As $T_1(G')$ and $T_2(G')$ are non-empty and G' is connected, $X' \neq \emptyset$. We claim that

$$r_1'(T_1(G') \cap X') + r_2'(T_2(G') \cap X') + |R \cap X'| - \mathrm{odd}_{G'}(X') \geq 1.$$

Clearly, this is true if $X' = \{v\}$. Suppose that $X := X' \backslash \{v\} \neq \emptyset$. Then $\mathrm{odd}_{G'}(X') \leq \mathrm{odd}_{G[S]}(X)$ and thus,

$$\begin{aligned} &r_1'(T_1(G') \cap X') + r_2'(T_2(G') \cap X') + |R(G') \cap X'| - \mathrm{odd}_{G'}(X') \\ &\quad \geq r_1(T_1 \cap X) + |R \cap X| - \mathrm{odd}_{G[S]}(X) \geq 1 \end{aligned}$$

by (b). This proves our claim. It now follows from Theorem 2 that there is an independent path-matching K of (G', M_1', M_2') having value $|R(G')| + 1$. By Lemma 15, K contains a path joining v and a vertex in T_1. Note that the path contains the edge uv. Hence, $K \backslash \{uv\}$ is an independent path-matching of (G, M_1, M_2) having value $|S \cap R|$. As $R(G') = S \cap R$, we have $\psi^K, \psi^{K \backslash \{uv\}} \in \mathcal{F}$ and $\psi^{K \backslash \{uv\}} = \psi^K - \chi^{uv}$. Since both ψ^K and $\psi^{K \backslash \{uv\}}$ must satisfy (9) with equality, we must have $c_{uv} = 0$.

Therefore, $c_e \neq 0$ implies that $e \in \gamma(S)$. We now show that, for all edges $e, f \in \gamma(S)$, we have $c_e = c_f$. Suppose that this is not the case. As $G[S \cap R]$ is connected, there exists a vertex $v \in S \cap R$ such that c_e takes on different values for edges in $\delta(v) \cap \gamma(S)$. Form the graph G' as follows. Take $G[S]$ and split v into v' and v'' such that all the edges $e \in \delta(v) \cap \gamma(S)$ for which c_e takes on the minimum value (of the edges in $\delta(v) \cap \gamma(S)$) are incident with v' and all of the others are incident with v''. Add a new vertex w and the edges wv' and wv''. Now, consider the independent path-matching problem on (G', M_1', M_2'), where $T_1(G') = T_1$, $T_2(G') = \{w\}$, $M_1' = M_1$ and M_2' is the free matroid. Let r_i' denote the rank function of M_i' for $i = 1, 2$. Notice that $|R(G')| = |S \cap R| + 1$. Let X' be a cut of G'. Observe that $X' \neq \emptyset$. We claim that

$$r_1'(T_1(G') \cap X') + r_2'(T_2(G') \cap X') + |R(G') \cap X'| - \mathrm{odd}_{G'}(X') \geq 1.$$

Clearly, this is true if $X' = \{w\}$. Assume that $X' \neq \{w\}$.

Let

$$\tilde{X} = \begin{cases} X' \backslash \{v', v'', w\} \cup \{v\} & \text{if } X' \cap \{v', v''\} \neq \emptyset; \\ X' \backslash \{w\} & \text{otherwise.} \end{cases}$$

Clearly, $\emptyset \neq \tilde{X} \subseteq S$. We see from (b) that, to prove the claim, it suffices to show that

$$r_1'(T_1(G') \cap X') + r_2'(T_2(G') \cap X') + |R(G') \cap X'| - \mathrm{odd}_{G'}(X') \geq r_1(T_1 \cap \tilde{X}) + |R \cap \tilde{X}| - \mathrm{odd}_{G[S]}(\tilde{X}).$$

Note that $r_1'(T_1(G') \cap X') = r_1(T_1 \cap \tilde{X})$, $r_2'(T_2(G') \cap X') = |X' \cap \{w\}|$, and $|R(G') \cap X'| \geq |R \cap \tilde{X}|$. Hence, it suffices to show that

$$\mathrm{odd}_{G'}(X') \leq \mathrm{odd}_{G[S]}(\tilde{X}) + |X' \cap \{w\}|. \tag{10}$$

We consider two cases.

Case 1: $X' \cap \{v', v''\} \neq \emptyset$.

Without loss of generality, assume that $v' \in X'$. If $w \notin X'$, then $\mathrm{odd}_{G'}(X') \leq \mathrm{odd}_{G[S]}(\tilde{X})$. (Indeed, if $v'' \in X'$, then, clearly, $\mathrm{odd}_{G'}(X') = \mathrm{odd}_{G[S]}(X)$; otherwise, it follows from the fact that, as $v''w \in E(G')$, every odd component of $G' - X'$ disjoint from $T_1(G') \cup T_2(G')$ cannot contain v'' and, therefore, is an odd component of $G - \tilde{X}$.) If $w \in X'$, then every odd component of $G' - X'$ not containing v'' and disjoint from $T_1(G') \cup T_2(G')$ is an odd component of $G - X$. It follows that $\mathrm{odd}_{G'}(X') \leq \mathrm{odd}_{G[S]}(X) + 1$. Hence, (10) holds.

Case 2: $X' \cap \{v', v''\} = \emptyset$.

If $w \notin X'$, then $\text{odd}_{G'}(X') \le \text{odd}_{G[S]}(X') = \text{odd}_{G[S]}(\tilde{X})$. If $w \in X'$, then observe that, if we identify the vertices v' and v'' in $G' - X'$, the number of odd components disjoint from $T_1(G') \cup T_2(G')$ cannot go down by more than 1. It follows that $\text{odd}_{G'}(X') \le \text{odd}_{G[S]}(X) + 1$. Hence, (10) holds. This completes the proof of our claim.

Applying Theorem 2 to (G', M'_1, M'_2), we see that there exists an independent path-matching K of (G', M'_1, M'_2) having value $|R(G')| + 1 = |S \cap R| + 2$. By Lemma 15, K contains a path joining w and a vertex in T_1. Furthermore, every vertex in $S \cap R(G')$ is incident with some edge in K. Now, K contains exactly one edge in $\{wv', wv''\}$. Without loss of generality, assume that $wv' \in K$. Let $pv' \neq wv'$ denote the other edge in K incident with v'. Let qv'' denote the edge in K incident with v''. (Note that pv' is not a matching edge and qv'' may or may not be a matching edge.) Let $K' = K \backslash \{wv', pv', qv''\}$. Let

$$K_1 = K' \cup \{qv\} \quad \text{and} \quad K_2 = \begin{cases} K' \cup \{pv, qv\} & \text{if } qv'' \text{ is a matching edge of } K, \\ K' \cup \{pv\} & \text{otherwise.} \end{cases}$$

Observe that K_1 and K_2 are independent path-matchings of (G, M_1, M_2) and $\psi^{K_1}, \psi^{K_2} \in \mathcal{F}$. Furthermore, $\psi^{K_1} = \psi^{K_2} + \chi^{qv} - \chi^{pv}$. Since $c_{pv} \neq c_{qv}$, ψ^{K_1} and ψ^{K_2} cannot both satisfy (9) with equality. This contradiction completes the proof. $\square$

2.5. Rank inequalities

We first prove two technical lemmas.

Lemma 17. *Let $i \in \{1, 2\}$ and $k > 0$ be an integer. Let $U \subseteq T_i$. Then G has an independent path-matching $K \subseteq \delta(U)$ having value k if and only if, for every subset A of U,*

$$r_i(A) + r_{3-i}(N(U \backslash A) \cap T_{3-i}) + |N(U \backslash A) \backslash T_{3-i}| \ge k.$$

Proof. By symmetry, it suffices to prove the lemma for $i = 1$. Consider the independent path-matching problem on (G', M'_1, M'_2) with $G' = G[\delta(U)]$, $T_1(G') = U$, $T_2(G') = N(U)$, $M'_1 = M_1 \backslash (T_1 \backslash U)$, and $M'_2 = M_2 \backslash (T_2 \backslash N(U)) \oplus M$, where M is the free matroid on $N(U) \backslash T_2$. Note that $R(G') = \emptyset$.

Observe that every independent path-matching of (G', M'_1, M'_2) having value k is an independent path-matching $K \subseteq \delta(U)$ of (G, M_1, M_2) having value k and conversely. By Theorem 2, there exists an independent path-matching of (G', M'_1, M'_2) having value k if and only if

$$\min_{X, \text{a cut}} (r'_1(U \cap X) + r'_2(N(U) \cap X)) \ge k.$$

Note that the minimum is attained by a cut X of the form $A \cup N(U \backslash A)$ for some $A \subseteq U$. To see this, let X' be any minimizer. Let $A = U \cap X'$ and let $\bar{X} = A \cup N(U \backslash A)$. Observe that $\bar{X}$ is a cut of G' and $\bar{X} \subseteq X'$. Hence, $r'_1(U \cap \bar{X}) + r'_2(N(U) \cap \bar{X}) \le r'_1(U \cap X') + r'_2(N(U) \cap X')$, implying that $\bar{X}$ is also a minimizer. Therefore, there exists an independent path-matching of (G', M'_1, M'_2) having value k if and only if $r'_1(A) + r'_2(N(U \backslash A)) \ge k$ for all $A \subseteq U$. As $r'_1(A) = r_1(A)$ and $r'_2(N(U \backslash A)) = r_2(N(U \backslash A) \cap T_2) + |N(U \backslash A) \backslash T_2|$ for all $A \subseteq N(U) \cap T_2$, the result follows. $\square$

Lemma 18. *Let $G = (V, E)$ be a bipartite graph with bipartition (T_1, T_2). Let M_1 and M_2 be loopless matroids on T_1 and T_2, respectively. Let $r = \min\{r_1(T_1), r_2(T_2)\}$, where r_i is the rank function of M_i for $i = 1, 2$. Suppose that, for every $i \in \{1, 2\}$,*

$$r_i(A) + \min\{r_i(T_i \backslash A), r_{3-i}(N(T_i \backslash A))\} \ge r$$

for every $A \subseteq T_i$ with equality only if $A = \emptyset$ or $A = T_i$. Then the inequality $x(E) \le r$ induces a facet of $\text{IP}(G, M_1, M_2)$.

Proof. We reduce the problem to a problem on the common independent set polytope via a trick. For $i = 1, 2$, define $r'_i : 2^E \to \mathbb{Z}$ as follows: for every $S \subseteq E$, $r'_i(S) = r_i(A)$, where $A = \{u \in T_i : S \cap \delta(u) \neq \emptyset\}$. Observe that r'_i is the rank function of the matroid M'_i on E derived from M_i by replacing each $v \in T_i$ with elements in $\delta(v)$ as parallel elements. Consider any $S \subseteq E$. Observe that S is an independent path-matching of (G, M_1, M_2) if and only if S is a common independent set of M'_1 and M'_2. Hence, $\text{IP}(G, M_1, M_2) = \text{CP}(M'_1, M'_2)$. To prove the lemma, it suffices to show that $x(E) \le r$ induces a facet of $\mathrm{CP}(M_1', M_2')$. Let $r'(S) = \min\{r_1'(S), r_2'(S)\}$. By Theorem 7, it suffices to show that there is no partition of E into proper subsets S_1 and S_2 with $r \ge r'(S_1) + r'(S_2)$.

Suppose that there exist non-empty $S_1, S_2 \subseteq E$ partitioning E such that $r \ge r'(S_1) + r'(S_2)$. Without loss of generality, assume that $r'(S_1) = r_1'(S_1)$. Let $A = \{u \in T_1 : S_1 \cap \delta(u) \ne \emptyset\}$. Note that $r_1'(S_1) = r_1(A)$, $r_1'(S_2) \ge r_1(T_1 \backslash A)$, and $r_2'(S_2) \ge r_2(N(T_1 \backslash A))$. Hence,

$$r \ge r'(S_1) + r'(S_2) \ge r_1(A) + \min\{r_1(T_1 \backslash A), r_2(N(T_1 \backslash A))\}.$$

As $A \ne \emptyset$, we must have $A = T_1$. However, as $S_2 \ne \emptyset$ and M_1', M_2' are loopless, we have $r'(S_2) > 0$. Hence, $r \ge r'(S_1) + r'(S_2) > r_1'(S_1) = r_1(A) = r_1(T_1) \ge r$, which is a contradiction. □

We are now ready to determine which rank inequalities are facet-inducing.

Theorem 19. *Let $i \in \{1, 2\}$. Let $U \subseteq T_i$. For each $B \subseteq N(U)$, define $f(B) := r_{3-i}(B \cap T_{3-i}) + |B \backslash T_{3-i}|$. The inequality*

$$x(\delta(U)) \le r_i(U) \tag{11}$$

is facet-inducing if and only if all of the following are satisfied:

(i) *U is closed (with respect to M_i);*

(ii) *for every non-empty proper subset A of U,*

$$r_i(A) + \min\{r_i(U \backslash A), f(N(U \backslash A))\} > r_i(U);$$

(iii) *for every non-empty proper subset B of $N(U)$,*

$$f(B) + \min\{f(N(U) \backslash B), r_i(N(N(U) \backslash B) \cap U)\} > r_i(U);$$

(iv) *$f(N(U)) \ge r_i(U)$ with equality only if $N(U) \subseteq T_{3-i}$, $\delta(N(U)) = \delta(U)$, and $N(U)$ is closed (with respect to M_{3-i}).*

Proof. By symmetry, we may assume that $i = 1$. We first show necessity. Suppose that (i) does not hold. Let $U' \subseteq T_1$ be such that U' is a proper superset of U and $r_1(U') = r_1(U)$. As G has no isolated vertices, $\delta(U') \ne \delta(U)$. Adding the rank inequality $x(\delta(U')) \le r_1(U')$ and the inequalities $-x_e \le 0$ for all $e \in \delta(U') \backslash \delta(U)$ gives (11). By Corollary 9, (11) is not facet-inducing.

Suppose that (ii) does not hold for some non-empty proper subset A of U. Note that $\delta(U) \ne \delta(A)$ and $\delta(U) \ne \delta(U \backslash A)$. If $r_1(A) + r_1(U \backslash A) = r_1(U)$, then adding the rank inequalities $x(\delta(A)) \le r_1(A)$ and $x(\delta(U \backslash A)) \le r_1(U \backslash A)$ gives (11). By Corollary 9, (11) is not facet-inducing. Suppose that $r_1(A) + f(N(U \backslash A)) \le r_1(U)$. As A is non-empty, $r_1(A) > 0$ and, thus, $f(N(U \backslash A)) < r_1(U)$. Let $B = N(U \backslash A) \cap T_2$ and $S = (N(U \backslash A) \backslash T_2) \cup T_1$. Let $C = \delta(U) \backslash (\delta(B) \cup \gamma(S))$. If $C = \emptyset$, then adding the rank inequality $x(\delta(B)) \le r_2(B)$, the cap inequality $x(\gamma(S)) \le |S \cap R|$, and the inequalities $-x_e \le 0$ for all $e \in (\delta(B) \cup \gamma(S)) \backslash \delta(U)$ gives $x(\delta(U)) \le f(N(U \backslash A))$, implying that (11) induces the empty face. Suppose that $C \ne \emptyset$. Then adding the rank inequalities $x(\delta(A)) \le r_1(A)$, $x(\delta(B)) \le r_2(B)$, the cap inequality $x(\gamma(S)) \le |S \cap R|$ and the inequalities $-x_e \le 0$ for all $e \in ((\delta(B) \cup \gamma(S)) \backslash \delta(U)) \cup (\delta(A) \cap (\delta(B) \cup \gamma(S)))$ gives the inequality $x(\delta(U)) \le d$ for some $d \le r_1(U)$. By Corollary 9, (11) is not facet-inducing.

Suppose that (iii) does not hold for some non-empty proper subset B of $N(U)$. Let $\bar{B} = N(U) \backslash B$. Let $B' = B \cap T_2$, $B'' = \bar{B} \cap T_2$, $S' = (B \backslash T_2) \cup T_1$, and $S'' = (\bar{B} \backslash T_2) \cup T_1$. Observe that $\delta(U)$ is equal to none of $\delta(B')$, $\delta(B'')$, $\gamma(S')$, and $\gamma(S'')$. If $f(B) + f(\bar{B}) \le r_1(U)$, then adding the rank inequalities $x(\delta(B')) \le r_2(B')$ and $x(\delta(B'')) \le r_2(B'')$, the cap inequalities $x(\gamma(S')) \le |S' \cap R|$ and $x(\gamma(S'')) \le |S'' \cap R|$, and the inequalities $-x_e \le 0$ for all $e \in (\delta(B \cap T_2) \cup \gamma(S') \cup \gamma(S'')) \backslash \delta(U)$ gives the inequality $x(\delta(U)) \le f(B) + f(\bar{B})$. By Corollary 9, (11) is not facet-inducing. Suppose that $f(B) + r_1(N(\bar{B}) \cap U) \le r_1(U)$. As $B \ne \emptyset$, $r_1(N(\bar{B}) \cap U) < r_1(U)$. It follows that $U \backslash (N(\bar{B}) \cap U) \ne \emptyset$. Adding the rank inequalities $x(\delta(B')) \le r_2(B')$, $x(\delta(N(\bar{B}) \cap U)) \le r_1(N(\bar{B}) \cap U)$, the cap inequality $x(\gamma(S')) \le |S' \cap R|$, and the inequalities $-x_e \le 0$ for all $e \in ((\delta(B') \cup \gamma(S)) \backslash \delta(U)) \cup (\delta((N(\bar{B}) \cap U)) \cap (\delta(B') \cup \gamma(S)))$ gives the inequality $x(\delta(U)) \le f(B) + r_1(N(\bar{B}) \cap U)$. By Corollary 9, (11) is not facet-inducing.

Suppose that (iv) does not hold. It is easy to see that, if $f(N(U)) < r_1(U)$, then (11) induces the empty face. Hence, assume that $f(N(U)) \ge r_1(U)$. If $f(N(U)) = r_1(U)$ and it is not true that $N(U) \subseteq T_2$ and $\delta(N(U)) = \delta(U)$, then adding the rank inequality $x(\delta(N(U) \cap T_2)) \le r_2(N(U) \cap T_2)$, the cap inequality $x(\gamma(S)) \le |S \cap R|$, where $S = (N(U) \backslash T_2) \cup T_1$, and the inequalities $-x_e \le 0$ for all $e \in \delta(N(U)) \backslash \delta(U)$ gives (11). By Corollary 9, (11) is not facet-inducing. If $f(N(U)) = r_1(U)$, $N(U) \subseteq T_2$, $\delta(N(U)) = \delta(U)$, and $r_2(U') = r_2(N(U))$ for some $U' \subseteq T_2$ such that U' is a proper superset of $N(U)$, then adding the rank inequality $x(\delta(U')) \leq r_2(U')$ and the inequalities $-x_e \leq 0$ for all $e \in \delta(U')\backslash\delta(U)$ gives (11). By Corollary 9, (11) is not facet-inducing.

We now show sufficiency. Let $\mathcal{F}$ be the set of independent path-matching vectors that satisfy (11) with equality. Let

$$\sum_{e\in E} c_e x_e \leq d \tag{12}$$

be a non-trivial valid inequality that is also satisfied by every element in $\mathcal{F}$ with equality. We show that (12) must be a scalar multiple of (11).

We first show that $c_{uv} = 0$ for every $uv \in E\backslash\delta(U)$.

Suppose that $f(N(U)) = r_1(U)$. Then, by (iv), $N(U) \subseteq T_2$, $\delta(N(U)) = \delta(U)$, and $N(U)$ is closed. It is easy to see from (ii) and (iv) that $r_1(A) + r_2(N(U\backslash A) \cap T_2) + |N(U\backslash A)\backslash T_2| = r_1(A) + f(N(U\backslash A)) \geq r_1(U)$ for all $A \subseteq U$. It follows from Lemma 17 that there exists an independent path-matching $K \subseteq \delta(U)$ of (G, M_1, M_2) having value $r_1(U)$. As both U and $N(U)$ are closed and uv is not incident with any vertex in $U \cup N(U)$, $K \cup \{uv\}$ is also an independent path-matching of (G, M_1, M_2). But $\psi^K, \psi^{K\cup\{uv\}} \in \mathcal{F}$ and they differ only in the uv-coordinate. As ψ^K and $\psi^{K\cup\{uv\}}$ must also satisfy (12) with equality, we have $c_{uv} = 0$.

Now, suppose that $f(N(U)) > r_1(U)$. Without loss of generality, assume that $u \in R \cup T_1\backslash U$ and $v \in R \cup T_2$. Consider the independent path-matching problem on (G', M_1, M_2'), where $G' = G - v$ and $M_2' = \begin{cases} M_2 & \text{if } v \in R, \\ M_2/v & \text{if } v \in T_2 \end{cases}$. Let r_2' denote the rank function of M_2'. We claim that, for every $A \subseteq U$,

$$r_1(A) + r_2'(N_{G'}(U\backslash A) \cap T_2(G')) + |N_{G'}(U\backslash A)\backslash T_2(G')| \geq r_1(U). \tag{13}$$

Clearly, (13) holds for $A = U$. Suppose that A is a proper subset of U. If $v \in T_2$, then

$$|N_{G'}(U\backslash A)\backslash T_2(G')| = |N_G(U\backslash A)\backslash T_2|$$

and

$$r_2'(N_{G'}(U\backslash A) \cap T_2(G')) = r_2(N_{G'}(U\backslash A) \cap T_2(G') \cup \{v\}) - 1 \geq r_2(N_G(U\backslash A) \cap T_2) - 1.$$

If $v \in R$, then

$$|N_{G'}(U\backslash A)\backslash T_2(G')| \geq |N_G(U\backslash A)\backslash T_2| - 1$$

and $r_2' = r_2$. In any case, we have

$$r_1(A) + r_2'(N_{G'}(U\backslash A) \cap T_2(G')) + |N_{G'}(U\backslash A)\backslash T_2(G')| \geq r_1(A) + f(N_G(U\backslash A)) - 1 \geq r_1(U)$$

by (ii) and (iv). This proves the claim. It now follows from Lemma 17 that there exists an independent path-matching $K \subseteq \delta(U)$ of (G', M_1, M_2') having value $r_1(U)$. (Note that K does not have any matching edge.) It is not difficult to see that K and $K \cup \{uv\}$ are independent path-matchings of (G, M_1, M_2) such that $\psi^K, \psi^{K\cup\{uv\}} \in \mathcal{F}$. As ψ^K and $\psi^{K\cup\{uv\}}$ must also satisfy (12) and they differ only in the uv-coordinate, we have $c_{uv} = 0$.

Thus, $c_e = 0$ for all $e \notin \delta(U)$. Consider the independent path-matching problem on (G', M_1', M_2'), where $G' = G[\delta(U)]$, $T_1(G') = U$, $T_2(G') = N(U)$, $M_1' = M_1\backslash(T_1\backslash U)$, and $M_2' = M_2\backslash(T_2\backslash N(U)) \oplus M$, where M is the free matroid on $N(U)\backslash T_2$. Observe that M_2' has rank function f. Note that, by Lemma 13, $\mathrm{IP}(G, M_1, M_2)|_{\delta(U)} = \mathrm{IP}(G', M_1', M_2')$. As $E(G') = \delta(U)$, and (ii), (iii), and (iv) hold, it follows from Lemma 18 that $x(\delta(U)) \leq r_1(U)$ induces a facet of $\mathrm{IP}(G', M_1', M_2')$. As $\mathrm{supp}(c) \subseteq \delta(U)$ and $\{x \in \mathrm{IP}(G', M_1', M_2') : x(\delta(U)) = r_1(U)\} \subseteq \{x \in \mathrm{IP}(G', M_1', M_2') : c^T x = d\}$, by Corollary 10, $c^T x \leq d$ is a positive scalar multiple of $x(\delta(U)) \leq r_1(U)$. □

3. Final remarks

One can see from the proofs that the non-facet-inducing inequalities among (3)–(7) either induce the empty face or can be written as integer linear combinations of other inequalities among (3)–(7). As (3)–(7) give a TDI system, the facet-inducing inequalities also give a TDI system.

Acknowledgements

The author is grateful to Bill Cunningham for helpful conversations, especially for pointing out the trick in Lemma 18.

The research in this paper is supported by NSERC PDF.

References

[1] W.J. Cook, W.H. Cunningham, W.R. Pulleyblank, A. Schrijver, Combinatorial Optimization, Wiley, New York, 1998.
[2] W.H. Cunningham, J.F. Geelen, The optimal path-matching problem, Combinatorica 17 (1997) 315–337.
[3] W.H. Cunningham, A.B. Marsh III, A primal algorithm for optimum matching, Math. Program. Stud. 8 (1978) 50–72.
[4] J. Edmonds, Maximum matching and a polyhedron with 0, 1 vertices, J. Res. Natl. Bur. Stand. B 69 (1965) 125–130.
[5] J. Edmonds, Submodular functions, matroids and certain polyhedra, in: R.K. Guy et al. (Eds.), Combinatorial Structures and their Applications, Gordon and Breach, New York, 1970, pp. 69–87.
[6] A. Frank, L. Szegő, Note on the path-matching formula, J. Graph Theory 41 (2002) 110–119.
[7] F.R. Giles, Submodular functions, graphs and integer polyhedra, Ph.D. Thesis, Department of Combinatorics and Optimization, University of Waterloo, Waterloo, Ontario, 1975.
[8] W.R. Pulleyblank, J. Edmonds, Facets of 1-matching polyhedra, in: C. Berge, D. Ray-Chaudhuri (Eds.), Hypergraph Seminar, Springer, Berlin, 1974, pp. 214–242.
[9] A. Schrijver, Combinatorial Optimization. Polyhedra and Efficiency, Springer-Verlag, Berlin, 2003.